Lecture 5

Image decomposition

Let $g = A f B$ (Separable). Then: $f = A^{-1} g B^{-1}$

(assume that A and B are invertible)

Write: $A^{-1} = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$; $B^{-1} = \begin{pmatrix} — & \vec{v}_1^{\,T} & — \\ — & \vec{v}_2^{\,T} & — \\ & \vdots & \\ — & \vec{v}_N^{\,T} & — \end{pmatrix}$

Then: $$g = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{u}_i \vec{v}_j^{\,T}}_{M_{N\times N}}$$

Proof: $$f = A^{-1} f B^{-1} = A^{-1} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} f_{ij} \begin{array}{cc} \begin{array}{ccc} & j^{th}\ \text{col} & \\ & \downarrow & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & 1 & 0 \\ 0 & \cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow i^{th}\ \text{row} \\ \\ \end{array} \end{array} \right) B^{-1}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} f_{ij} \; A^{-1} \begin{pmatrix} 0 & \cdots 0 & 0 \\ \vdots & \vdots & \\ 0 & \cdots 1 \cdots & 0 \\ \vdots & \vdots & \\ 0 & 0 \cdots & 0 \end{pmatrix} B^{-1} = \underbrace{\vec{u}_i \, \vec{v}_j^{\,T}}_{N\times N \text{ matrix}}$$

$\therefore f =$ linear combination of $\{\vec{u}_i \vec{v}_j^{\,T}\}_{i,j}$

Definition: Each $\vec{u}_i v_j^T$ is called an <u>elementary image</u>.

$\vec{u}_i v_j^T$ is also called the <u>outer product</u> of $\vec{u}_i$ and $\vec{v}_j$.

One important task in image processing:

Choose A and B such that:

1. Transformed image requires less storage (Many $g_{ij} = 0$)
2. Take away some terms $g_{ij} \vec{u}_i \vec{v}_j^T$ (e.g. high-frequency) $\rightarrow$ Better image!!
3. A^{-1} and B^{-1} are easy to compute!

<u>Common example:</u>

Orthogonal $U \Leftrightarrow U^T U = I \quad \therefore U^{-1} = U^T$.

Example: Let $A = \underbrace{\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}}_{A^{-1}}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}}_{g}\underbrace{\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}}_{B^{-1}}$

Then:

$$A = \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\left[1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + 2\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}$$

$$= 1\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix} + 2\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}$$

$$= \underset{g_{11}}{\underset{\parallel}{1}}\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}3 & 1\\ 0 & 0\end{pmatrix} + \underset{g_{22}}{\underset{\parallel}{2}}\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 3\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}3 & 1\\ 0 & 0\end{pmatrix} = \underbrace{\begin{pmatrix}1\\ 1\end{pmatrix}}_{\vec{u}_1}\underbrace{\begin{pmatrix}3, & 1\end{pmatrix}}_{\vec{v}_1^{\,T}} \qquad \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 3\end{pmatrix} = \underbrace{\begin{pmatrix}2\\ 1\end{pmatrix}}_{\vec{u}_2}\underbrace{\begin{pmatrix}1 & 3\end{pmatrix}}_{\vec{v}_2^{\,T}}$$

$$= 1\underbrace{\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix}}_{\text{elementary image}} + 2\underbrace{\begin{pmatrix}2 & 6\\ 1 & 3\end{pmatrix}}_{\text{elementary image}}$$

Image decomposition

Image decomposition based on Singular Value Decomposition (SVD)

Definition: (SVD) For any $g \in M_{m\times n}$, the singular value decomposition (SVD) of g is a matrix factorization: $g = U \Sigma V^T$, where $U \in M_{m\times m}$, $V \in M_{n\times n}$ are orthogonal, $\Sigma \in M_{m\times n}$ is a diagonal matrix ($\Sigma_{ij} = 0$ if $i \neq j$) with diagonal entries given by: $\underbrace{\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r}_{\text{singular values}} > 0$ with $r \leq \min(m, n)$. ($UU^T = U^TU = I$; $VV^T = V^TV = I$)

Theorem: The rank of g is given by the number of non-zero singular values.

Proof: Rank = dim of column space.

Recall that $\operatorname{rank}(AB) = \operatorname{rank}(B)$ if A is invertible

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Suppose $g = U\Sigma V^T$. Since U and V are invertible, $\operatorname{rank}(g) = \operatorname{rank}(\Sigma)$

$= \#$ of non-zero singular values

Remark: Consider an image g. Let $g = U \Sigma V^T$ be the SVD of g (with diagonal entries of Σ given by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$)

1. Note that
$$g = U \Sigma V^T = \sum_{i=1}^{r} \sigma_i \, U \begin{array}{cc} \begin{array}{c} i\text{th} \\ \downarrow \end{array} & \\ \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} & \leftarrow i\text{th} \end{array} V^T = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

$\vec{u}_i \vec{v}_i^T$ is called the eigen-image of g under SVD.

2. For $N \times N$ image, the required storage is:
$$(\underbrace{N}_{\vec{u}_i} + \underbrace{N}_{\vec{v}_i} + \underbrace{1}_{\sigma_i}) \times \underbrace{r}_{r \text{ terms}} = (2N+1)r$$

Observation about SVD

Let $A = U \Sigma V^T$ (Let $A \in M_{n\times n}$)

Write $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{pmatrix}$; $V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$; $\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix}$

- $A^T A = (U \Sigma V^T)^T (U \Sigma V^T) = V \Sigma^T \underbrace{U^T U}_{I} \Sigma V^T = V \Sigma^T \Sigma V^T$

$$\Rightarrow (A^T A) V = V \begin{pmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{pmatrix} \Rightarrow \begin{pmatrix} | & & | \\ (A^T A)\vec{v}_1 & \cdots & (A^T A)\vec{v}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \sigma_1^2 \vec{v}_1 & \cdots & \sigma_n^2 \vec{v}_n \\ | & & | \end{pmatrix}$$

$$\Rightarrow (A^T A)\vec{v}_1 = \sigma_1^2 \vec{v}_1, \ \ldots, \ (A^T A)\vec{v}_n = \sigma_n^2 \vec{v}_n$$

$\therefore$ $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are eigenvectors of $A^T A$ with eigenvalues $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$.

- $AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma \underbrace{V^T V}_{I} \Sigma^T U^T = U\Sigma\Sigma^T U^T$

$$\Rightarrow (AA^T)U = U\begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{pmatrix} \Rightarrow \begin{pmatrix} | & & | \\ AA^T\vec{u}_1 & \cdots & AA^T\vec{u}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \sigma_1^2\vec{u}_1 & \cdots & \sigma_n^2\vec{u}_n \\ | & & | \end{pmatrix}$$

$\therefore$ $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ are eigenvectors of AA^T with eigenvalues $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$

- $A = U\Sigma V^T \Rightarrow AV = U\Sigma \Rightarrow \begin{pmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \sigma_1\vec{u}_1 & \sigma_2\vec{u}_2 & \cdots & \sigma_n\vec{u}_n \\ | & | & & | \end{pmatrix}$

$\therefore$ For $\sigma_1, \sigma_2, \ldots, \sigma_r > 0$,

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1}, \quad \vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2}, \quad \ldots, \quad \vec{u}_r = \frac{A\vec{v}_r}{\sigma_r}$$

We can obtain $\vec{u}_1, \ldots, \vec{u}_r$ from $\vec{v}_1, \ldots, \vec{v}_r$.

- $U^T U = I \Rightarrow \vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

$V^T V = I \Rightarrow \vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

$\therefore \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ are orthonormal.

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ are orthonormal.

How to compute SVD

Let $A \in M_{n\times n}$

Step 1: Find eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of $A^T A \in M_{n\times n}$ (with $\|\vec{v}_j\| = 1$, $j = 1, \dots, n$)

[Recall: $(A^T A)\vec{v}_j = \lambda_j \vec{v}_j$]

Step 2: Define:

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix} \in M_{n\times n}$$

Step 3: For non-zero $\sigma_1, \sigma_2, \dots, \sigma_r$,

let $\vec{u}_1 = \dfrac{A\vec{v}_1}{\sigma_1}$, $\vec{u}_2 = \dfrac{A\vec{v}_2}{\sigma_2}$, $\dots$, $\vec{u}_r = \dfrac{A\vec{v}_r}{\sigma_r}$

Step 4: Extend $\{\vec{u}_1, \dots, \vec{u}_r\}$ to the basis $\{\vec{u}_1, \dots, \vec{u}_r, \dots, \vec{u}_n\}$ of $\mathbb{R}^n$

Step 5: Let:

$$U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{pmatrix} \in M_{m\times m}$$

$$V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}$$

Then: $A = U \Sigma V^T$

Example: (2×2 example) Find the SVD of $A = \begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix}$

Step 1: $A^TA = \begin{pmatrix} 25 & -15 \\ -15 & 25 \end{pmatrix}$

Characteristic polynomial : $\det(A^TA - \lambda I) = (25-\lambda)(25-\lambda) - 15^2$

$$= \lambda^2 - 50\lambda + 400$$

$$= (\lambda - 10)(\lambda - 40)$$

$\therefore$ A^TA has two eigenvalues : $\lambda = 10$ and $\lambda = 40$

For $\lambda = 40$, $(A^TA - 40I) = \begin{pmatrix} -15 & -15 \\ -15 & -15 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. $\vec{v} = x_1 \begin{pmatrix} -1 \\ +1 \end{pmatrix}$

Choose $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} / \sqrt{2} = \begin{pmatrix} -1/\sqrt{2} \\ +1/\sqrt{2} \end{pmatrix}$

For $\lambda = 10$, $(A^TA - 10I) = \begin{pmatrix} 15 & -15 \\ -15 & 15 \end{pmatrix}$

Find null space to find eigenvector.

RREF of $\begin{pmatrix} 15 & -15 \\ -15 & 15 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = 0$

Let $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ = eigenvector

Then: $x_1 - x_2 = 0$

$\therefore \vec{v} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Choose $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{1^2 + 1^2} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$$\therefore \quad V = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{pmatrix}$$

<u>Step 2:</u>

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \begin{pmatrix} -1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} \quad \text{and} \quad \vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2} = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}.$$

$\therefore$ SVD of A is:

$$\begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix} = \underbrace{\begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{V^T}$$

Definition: For any k $(0 \leq k \leq r)$, we define

$$g_k = \sum_{j=1}^{k} \sigma_j \vec{u}_j \vec{v}_j^{\,T} \quad \text{(rank-}k\text{ approximation of } g\text{)}$$

$$= \underbrace{U \begin{pmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_k & \\ & & & & \ddots \end{pmatrix} V^T}_{\text{Rank-}k\text{ !!}}$$